The approximation numbers of Hardy–type operators on trees.

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Abstract

The Hardy operator T_a on a tree Γ is defined by

$$(T_a f)(x) := v(x) \int_a^x u(t) f(t) dt \qquad \text{for } a, x \in \Gamma.$$

Properties of T_a as a map from $L^p(\Gamma)$ into itself are established for $1 \leq p \leq \infty$. The main result is that, with appropriate assumptions on u and v, the approximation numbers $a_n(T_a)$ of T_a satisfy

$$(*) \lim_{n\to\infty} n a_n(T_a) = \alpha_p \int_\Gamma |uv| dt$$

for a specified constant α_p and $1 < p < \infty$. This extends results of Naimark, Newman and Solomyak for $p = 2$. Hitherto, for $p \neq 2$, (*) was unknown even when Γ is an interval. Also, upper and lower estimates for the l^q and weak-l^q norms of $\{a_n(T_a)\}$ are determined.

1 Introduction.

In [1],[2]and [6] results were established for the Hardy operator

$$Tf(x) = v(x) \int_0^x f(t) u(t) dt \tag{1. 1}$$

as a map from $L^p(0,\infty)$ to $L^p(0,\infty)$, for $1 \leq p \leq \infty$. When $p \in (1,\infty)$, it was proved in [2] that under appropriate conditions on u and v the approximation numbers $a_n(T)$ of T satisfy

$$\lim_{n\to\infty} n a_n(T) = \frac{1}{\pi} \int_0^\infty |u(t)v(t)| dt \tag{1. 2}$$

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when p=2, and when $1 < p < \infty$

$$\begin{aligned}\frac{\alpha_p}{4}\int_0^\infty |u(t)v(t)|dt &\leq \liminf_{n\to\infty} na_n(T) \\ &\leq \limsup_{n\to\infty} na_n(T) \leq \alpha_p \int_0^\infty |u(t)v(t)|dt \qquad (1.\ 3)\end{aligned}$$

for a specified constant α_p depending on p. For the cases $p=\infty$ and $p=1$ similar estimates were derived in [6] but with $v_s(t) = \lim_{\varepsilon\to 0}\|v\|_{\infty,(t-\varepsilon,t+\varepsilon)}$ instead of $v(t)$ when $p=\infty$ and $u_s(t)$ instead of $u(t)$ in the case $p=1$. Furthermore, in [2] and [6] two-sided estimates are given for the l^α and weak-l^α norms of the sequence of approximation numbers in the case when the Hardy operator is compact.

A special case of the main result in this paper is that the counterpart of (1.2) in $L^p(0,\infty)$, namely

$$\lim_{n\to\infty} na_n(T) = \alpha_p \int_0^\infty |u(t)v(t)|dt, \qquad (1.\ 4)$$

holds for *all* $p \in (1,\infty)$; the general result is that the analogue of (1.4) when the interval $(0,\infty)$ is replaced by a tree Γ is true. Such Hardy operators on trees have already been investigated in [5] where it was shown that they occur naturally in spectral problems defined on domains with irregular boundaries. Necessary and sufficient criteria for the boundedness of Hardy operators between various Lebesgue spaces on Γ are established in [7], but the complex nature of the problem is such that the neat abstract result is difficult to apply even for the most elementary of trees. It is therefore to be expected that the problems of compactness and estimating the approximation numbers are likely to be much more complicated than in the interval case. This, confirmed in this paper, is what makes it so surprising that the analogue of (1.4) for a tree is established here when $p \neq 2$ before it was known for an interval. Estimates for l^q and weak-l^q norms of the approximation numbers of T are also obtained.

In [9] the case $p=2$ of the problem subsequently studied in [2] and [6] for general $p \in [1,\infty]$ was considered and (1.2) proved, using Hilbert space methods which do not extend to general values of p. The same problem on a tree Γ is the subject of [8] where an intensive study is made of problems on trees which are closely related to those here, but in the case $p=2$ only, and using methods which are very different from those in this paper. The conditions imposed to ensure the validity of the analogue of (1.4) for a tree in [8] are similar to those here, but a comparison seems difficult in general (see Remark 6.12). The main difference is that in [8] they relate to arbitrary partitions of Γ into intervals, whereas our partitions are into connected subsets specifically determined by functions which have a fundamental role in the analysis.

2 Preliminaries.

In this section we recall the definition of a tree Γ, introduce a Hardy–type operator on the tree and quote from [7] the criterion for the boundedness of the operator as a map from $L^p(\Gamma)$ into $L^p(\Gamma)$.

A tree Γ is a connected graph without loops or cycles, where the edges are non-degenerate closed line segments whose end–points are the vertices. Each vertex of Γ is of finite degree, i.e. only a finite number of edges emanate from each vertex. For every $x, y \in \Gamma$ there is a unique polygonal path in Γ which joins x and y. The distance between x and y is defined to be the length of this polygonal path and in this way Γ is endowed with a metric topology.

Lemma 2.1 *Let $\tau(\Gamma)$ be the metric topology on Γ. Then*

(i) *a set $A \subset \Gamma$ is compact if and only if it is closed and meets only a finite number of edges;*

(ii) *$\tau(\Gamma)$ is locally compact;*

(iii) *Γ is the union of a countable number of edges. Thus if Γ is endowed with the natural 1-dimensional Lebesgue measure it is a σ-finite measure space.*

Proof. See [5]. $\square$

Let $x, y \in \Gamma$ and denote by (x, y) the unique path joining x, y in Γ. For $a \in \Gamma$ we define $t \succeq_a x$ (or $x \preceq_a t$) to mean that x lies on the path (a, t). We write $x \prec_a t$ for $x \preceq_a t$ and $x \neq t$. This is a partial ordering on Γ and the ordered graph so formed is referred to us a tree rooted at a and denoted by Γ_a. If a is not a vertex we make it one by replacing the edge on which it lies by two edges. In this way Γ_a is the unique finite union of subtrees $\Gamma_{a,i}$ which intersect only at a.

Note that if $x \notin (a, b)$ then $x \preceq_a y$ if and only if $x \preceq_b y$.

We shall use the following notation. For a subtree K of Γ, $V(K)$, $E(K)$ will denote respectively the sets of vertices, edges of K and ∂K will denote the set of boundary points of K in Γ. The notation $K \subset\subset \Gamma$ will be used to mean that the closure of K is a compact subset of Γ; note that, from Lemma 2.1 (i) this implies that K meets only a finite number of edges of Γ. The characteristic function of a set E will be denoted by χ_E. The integral is interpreted in the following sense :

$$\int_E g = \sum_e \int_{e \cap E} g$$

where

$$\int_{e \cap E} g = \int_c^d g(x)\chi_E(x)dx,$$

the integral $\int_c^d$ being over the set of points lying in the path (c, d). For a measurable subset K of Γ we define the norm

$$\|f\|_{p,K} = \big(\int_K |f|^p \big)^{1/p}$$

on $L^p(K)$. The $L^p(\Gamma)$ norm will be denoted by $\|\cdot\|_p$ if there is no chance of confusion. Also, if the value of p is clear from the context, we shall write $\|\cdot\|_K, \|\cdot\|$ for the L^p norms on K, Γ respectively. If A is a bounded map between normed spaces X, Y we denote its norm by

$$\|A|X \to Y\|$$

This will be simplified to $\|A\|$ if the spaces X, Y are unambiguous.

A connected subset of Γ is a subtree if we add its boundary points to the set of vertices of Γ, and hence form new edges from existing ones. Hereafter we shall always adopt this convention when we refer to subtrees.

Definition 2.2 *Let K be a subtree of Γ containing a. A point $t \in \partial K$ is said to be maximal if every $x \succ_a t$ lies in $\Gamma \setminus K$. We denote by $\mathbf{I}_a(\Gamma)$ (or simply $\mathbf{I}_a$) the set of subtrees K of Γ containing a whose boundary points are all maximal.*

We assume throughout, unless mentioned otherwise, that u, v satisfy the following conditions:

$$u \in L^{p'}(K),\ v \in L^p(\Gamma), \quad \text{for every } K \subset\subset \Gamma. \tag{2. 1}$$

We may assume, without loss of generality, that $u, v \geq 0$. This is because multiplication by $\operatorname{sgn} u$ and $\operatorname{sgn} v$ are isometries on $L^p(\Gamma)$; recall that $\operatorname{sgn} u = u/|u|$ when $u \neq 0$ and 1 otherwise.

Definition 2.3 *Let Γ be a tree, $1 \leq p \leq \infty$, and let u and v be measurable functions on Γ which satisfy (2. 1). For $x \in \Gamma$ and $f \in L^p(\Gamma)$ we define the Hardy operator by*

$$T_a f(x) := v(x) \int_a^x f(t)u(t)dt, \quad a \in \Gamma. \tag{2. 2}$$

In [7] the following necessary and sufficient condition for the boundedness of T_a was obtained.

Theorem 2.4 *Let $1 \leq p \leq \infty$, $a \in \Gamma$, and suppose u and v satisfy (2. 1). For $K \in \mathbf{I}_a$ define*

$$\alpha_K := \inf\{\|f\|_p : \int_a^t |f||u| = 1 \ \textit{ for all } \ t \in \partial K\}. \tag{2. 3}$$

Then T_a is bounded from $L^p(\Gamma)$ into $L^p(\Gamma)$ if and only if

$$A := \sup_{K \in \mathbf{I}_a} \frac{\|v\chi_{\Gamma \setminus K}\|_p}{\alpha_K} < \infty. \tag{2. 4}$$

Moreover, $A \leq \|T_a\| \leq 4A$.

3 Bounds for the approximation numbers

We recall that, given any $m \in \mathbf{N}$, the m–th approximation number of a bounded operator $T : L^p(\Gamma) \to L^p(\Gamma)$, $a_m(T)$, is defined by

$$a_m(T) := \inf \|T - F|L^p(\Gamma) \to L^p(\Gamma)\|,$$

where the infimum is taken over all bounded linear maps $F : L^p(\Gamma) \to L^p(\Gamma)$ with rank less than m.

A measure of non-compactness of T is given by

$$\beta(T) := \inf \|T - P|L^p(\Gamma) \to L^p(\Gamma)\|,$$

where the infimum is taken over all compact linear maps $P : L^p(\Gamma) \to L^p(\Gamma)$. Since $L^p(\Gamma)$ has the approximation property for $1 \leq p \leq \infty$, T is compact if and only if $a_m(T) \to 0$ as $m \to \infty$, and $\beta(T) = \lim_{n\to\infty} a_n(T)$.

Definition 3.1 *Let* K *be a subtree of* Γ *and* $a \in \Gamma$. *We define:*

$$A(K) \equiv A(K,u,v) := \begin{cases} \sup_{f \in L^p(K), f \neq 0} \inf_{\alpha \in \mathbf{C}} \frac{\|T_{a,K} f - \alpha v\ \|_{p,K}}{\|f\|_{p,K}} & \textit{if}\ \ \mu(K) > 0, \\ 0 & \textit{if}\ \ \mu(K) = 0, \end{cases}$$

where

$$T_{a,K} f(x) := v(x)\chi_K(x) \int_a^x u(t) f(t) \chi_K(t) dt,$$

and

$$\mu(K) := \begin{cases} \int_K |v(t)|^p dt & 1 \leq p < \infty, \\ \int_K |v(t)| dt & p = \infty. \end{cases}$$

Lemma 3.2 *The number* $A(K,u,v)$ *in Definition 3.1 is independent of* $a \in \Gamma$.

Proof. Denote by S the canonical map of $L^p(K)$ into its quotient by the space of scalar multiples of v. Then $A(K) = \|ST_{a,K}|L^p(K) \to L^p(K)\|$. For $b \in \Gamma$ we have $T_{b,K} f = v\chi_K \int_b^a f u \chi_K dt + T_{a,K} U f$, where $Uf(t) = -f(t)$ if t lies on the path (a,b) and $f(t)$ otherwise. Clearly U is a linear isometry of $L^p(K)$ onto itself and $ST_{b,K} = ST_{a,K} U$. $\square$

Corollary 3.3 *For all subtrees $K \subseteq \Gamma$*

$$A(K) \leq \inf_{a \in \Gamma} \|T_{a,K}|L^p(K) \to L^p(K)\|.$$

Note that if Λ is a subtree of Γ, $a \in \Gamma$ and b the nearest point of Λ to a then $T_{b,\Lambda} = T_{a,\Lambda}$ and $\|T_{b,\Lambda}\| := \|T_{b,\Lambda}|L^p(\Lambda) \to L^p(\Lambda)\| \leq \|T_a\| =: \|T_a|L^p(\Gamma) \to L^p(\Gamma)\|$. Moreover if $\Lambda' \subset \Lambda$, with c the nearest point of Λ' to a and (b,c) a subinterval of an edge of Λ then $T_{b,\Lambda}f = T_{b,\Lambda}(f\chi_{(b,c)}) + T_{c,\Lambda'}(f\chi_{\Lambda'})$, whence $0 \leq \|T_{b,\Lambda}\| - \|T_{c,\Lambda'}\| \leq \|u\|_{p',(b,c)}\|v\|_{p,(b,c)}$. This remark yields

Lemma 3.4 *For $1 \leq p \leq \infty, \|T_{x,K}|L^p(K) \to L^p(K)\|$ is continuous in x.*

Let $x \in \Gamma$. Denote by $\Gamma_{x,i}$ $i = 1, \dots, n_x$ the non-overlapping subtrees of Γ which are the closures of the connected components of $\Gamma \setminus \{x\}$, and set $T_{x,i} \equiv T_{x,\Gamma_{x,i}}$ and $\|T_{x,i}\| \equiv \|T_{x,i}|L^p(\Gamma_{x,i}) \to L^p(\Gamma_{x,i})\|$. We suppose that the numbering is done in the order of descending norms of the $T_{x,i} : L^p(\Gamma_{x,i}) \to L^p(\Gamma_{x,i})$. Clearly $\|T_x\| = \max_{i=1,\dots,n_x} \|T_{x,i}\|$.

Call a point $x \in \Gamma$ *simple* if there is just one $T_{x,i}$ with maximal norm, so that $\|T_{x,1}\| > \|T_{x,2}\|$. If a is a simple point and (a,y) the first edge of $\Gamma_{a,1}$ then by continuity either there is a point z of (a,y) which is not simple or $a \notin \Gamma_{y,1}$. If the latter, continue the path beginning with (a,y) along the initial edge of $\Gamma_{y,1}$. By induction thus define a path l in Γ satisfying one of the following:

(i) l is finite and its end b is not simple;

(ii) l is finite, its end b is simple and $\{x : x \succeq_a b\} = \emptyset$;

(iii) l is infinite.

Now (ii) is impossible since $\lim_{x \to b} \|T_{x,1}\| = 0$, and $\|T_{x,1}\| \geq A(\Gamma)$. Also (iii) implies T is not compact. For if x is in l, $\|T_{x,1}\| \geq A(\Gamma)$ and hence there is a compact subset K of $\Gamma_{x,1}$ and a function f supported in K with $\|f\| \leq 1$ and $\|T_a f\|_K \geq \frac{1}{2}A(\Gamma)$. It follows that there is a sequence of disjoint compact sets K_n and functions f_n with the same property. Then, if $m > n, \|T_a(f_n - f_m)\|_\Gamma \geq \|T_a f_n\|_{K_n} \geq \frac{1}{2}A(\Gamma)$. Thus, if T_a is compact, (i) holds. Moreover, $\|T_b\| = \min_{x \in \Gamma} \|T_x\|$. For if $x \neq b$ then $x \notin$ one of $\Gamma_{b,1}$, $\Gamma_{b,2}$, say $\Gamma_{b,2}$. Then if $\Gamma_{x,j}$ is the subtree containing b, $\|T_{x,j}\| \geq \|T_{b,2}\| = \|T_{b,1}\|$. From this we have the following result which will be an important tool for determining a lower bound for $A(K)$ once Theorem 3.8 below is available.

Lemma 3.5 *Suppose T_a is compact and that there exist $i \neq j$ such that $\|T_{a,i}\|, \|T_{a,j}\| \leq \|T_a\|$. Then*

$$\min\{\|T_{a,i}\|, \|T_{a,j}\|\} \leq \min_{x \in \Gamma} \|T_x\|.$$

Proof. The result is a consequence of the discussion preceding the lemma if a is not simple. If a is simple, then, with b the non-simple end-point of the path l in (i) above, and $\min\{\|T_{a,i}\|, \|T_{a,j}\|\} = \|T_{a,i}\|$, say, we have $\|T_{a,i}\| < \|T_a\|$ and $\Gamma_{a,i}$ is a subtree of some tree $\Gamma_{b,k}$. Thus

$$\|T_{a,i}\| \leq \|T_{b,k}\| \leq \|T_b\| = \min_{x\in\Gamma} \|T_x\|.$$

□

In the next two lemmas $\|\cdot\|_{p,\mu}$ denotes the norm in $L^p(\Gamma, d\mu)$, where $d\mu(t) = |v(t)|^p dt$.

Lemma 3.6 *If $1 < p \leq \infty$ there is a unique scalar c_f such that $\|f - c_f e\|_{p,\mu} = \inf_{c\in\mathbf{C}} \|f - ce\|_{p,\mu}$ for $e \neq 0, e \in L^p(\Gamma, d\mu)$.*

Proof. Since $\|f - ce\|_{p,\mu}$ is continuous in c and tends to ∞ as $c \to \infty$, the existence of c_f is guaranteed by the local compactness of $\mathbf{C}$. For $1 < p < \infty$ the uniqueness follows from the uniform convexity of $L^p(\Gamma, d\mu)$. Let $p = \infty$, and suppose that there are two values of $c_f, c_1 \neq c_2$. This yields the contradiction $\|f - (1/2)(c_1 + c_2)\|_{p,\mu} < \|f - c_1\|_{p,\mu}$.□

Lemma 3.7 *The map $f \to c_f : L^p(\Gamma, d\mu) \to \mathbf{C}$ is continuous for $1 < p \leq \infty$.*

Proof. Suppose that $c_{g_n} \to c$ as $g_n \to f$. Then

$$\|g_n - c_f\|_{p,\mu} \geq \|g_n - c_{g_n}\|_{p,\mu}$$

and so

$$\|f - c_f\|_{p,\mu} \geq \|f - c\|_{p,\mu}$$

which gives $c = c_f$ □

Theorem 3.8 *Let $1 < p \leq \infty$. If T_a is compact $A(\Gamma) = \min_{x\in\Gamma} \|T_x|L^p(\Gamma) \to L^p(\Gamma)\|$.*

Proof. There is a non-simple point b at which $\|T_x\|$ attains its minimum. If $\alpha < \|T_b\|$ there exist $f_i, i = 1, 2$, supported in $\Gamma_{b,i}$ with $\|f_i\| = 1, \|T_b f_i\| > \alpha$, and f_1 positive, f_2 negative. Clearly the same is true of the corresponding values of c_f, say c_1, c_2. Then by continuity there is a $\lambda \in [0, 1]$ such that $c_g = 0$ for $g = \lambda f_1 + (1 - \lambda) f_2$, and $\|T_b g\|^p = \lambda^p \|T_b f_1\|^p + (1 - \lambda)^p \|T_b f_2\|^p > \alpha^p \|g\|^p$. Then, by Lemma 3.6,

$$A(\Gamma) \geq \inf_c \|(T_b - cv)g\|/\|g\| = \|T_b g\|/\|g\| > \alpha.$$

Since $\alpha < \|T_b\|$ is arbitrary, $A(\Gamma) \geq \|T_b\|$ and the result follows from Corollary 3.3.□

The next lemma establishes an important geometrical property of a tree which is an essential ingredient of the subsequent analysis. First we make some observations.

Suppose w is a non–negative function defined on the set of all closed subtrees of a tree Γ, satisfying

$$X \subseteq Y \Rightarrow w(X) \leq w(Y). \tag{3. 1}$$

Define

- $$N_\varepsilon(\Gamma) = \min_{\mathcal{F} \in \mathcal{S}_\varepsilon(\Gamma)} \#\mathcal{F}$$

  where $\mathcal{S}_\varepsilon := \{\mathcal{F}; \mathcal{F}$ is a set of non–overlapping closed subtrees of Γ such that $i) \cup_{X \in \mathcal{F}} X = \Gamma, ii) X \in \mathcal{F} \Rightarrow w(X) \leq \varepsilon\}$;

- $$M_\varepsilon(\Gamma) = \max_{\mathcal{G} \in \mathcal{L}_\varepsilon(\Gamma)} \#\mathcal{G}$$

  where $\mathcal{L}_\varepsilon := \{\mathcal{G}; \mathcal{G}$ is a set of non–overlapping closed subtrees of Γ such that i) $\cup_{X \in \mathcal{G}} X = \Gamma$ ii) $\#\{X; X \in \mathcal{G}, w(X) \leq \varepsilon\} \leq 1\}$

Two non–overlapping closed subtrees of Γ can have at most one point in common, for otherwise Γ would contain a cycle. A *chain* $\mathcal{C}$ of closed subtrees is a sequence $X_1, \ldots, X_l$ of closed subtrees such that $X_i \cap X_{i+1} = \{x_i\}$ $(i = 1, \ldots, l-1)$ where the x_i are distinct. The *length* of $\mathcal{C}$ is l.

There is a set $\mathcal{F} \in \mathcal{S}_\varepsilon(\Gamma)$ with $\#\mathcal{F} = N_\varepsilon(\Gamma)$ (possibly ∞). Let $\mathcal{C}$ be a chain of elements of $\mathcal{F}$ of maximal length l. Then we have the following:

(i) If $l = 1$ then $\#\mathcal{F} = 1$ and so $w(\Gamma) \leq \varepsilon$.

(ii) If $l = 2$ define $Y = X_1 \cup X_2$. Then if $\Gamma \neq Y$, $\Gamma \setminus Y^o$ is a closed subtree and $N_\varepsilon(\Gamma \setminus Y^o) = N_\varepsilon(\Gamma) - 2$. Moreover Y is a closed subtree of Γ and $w(Y) > \varepsilon$. For since l is maximal x_1 lies in every member of $\mathcal{F}$ so that $\Gamma \setminus Y^o$ is a closed subtree. Also $\Gamma \setminus Y^o = \cup_{X \in \mathcal{F}'} X$, where $\mathcal{F}' = \mathcal{F} \setminus \{X_1, X_2\}$, which implies that $N_\varepsilon(\Gamma \setminus Y^o) \leq N_\varepsilon(\Gamma) - 2$. If $N_\varepsilon(\Gamma \setminus Y^o) < N_\varepsilon(\Gamma) - 2$, there exists $\mathcal{F}''$, a suitable covering of $\Gamma \setminus Y^o$ with $\#\mathcal{F}'' = N_\varepsilon(\Gamma \setminus Y^o)$, and then $\mathcal{F}'' \cup \{X_1, X_2\} \in \mathcal{S}_\varepsilon(\Gamma)$ which contradicts the definition of $N_\varepsilon(\Gamma)$. Finally $w(Y) > \varepsilon$ for if not, on taking $\mathcal{F}''' = \mathcal{F}' \cup \{Y\}$ we have a contradiction.

(iii) If $l \geq 3$, suppose $Z_1, \ldots, Z_k$ are the sets in $\mathcal{F}$ which meet X_{l-1} in a point different from x_{l-2}. Then since l is maximal $X_1, X_2, \ldots, X_{l-1}, Z_i$, is a chain of maximal length for $i = 1, 2, \ldots, k$. Either a) $k = 1$ or b) $k > 1$. If a), we have $Z_i = X_l$ and we take $Y = X_l \cup X_{l-1}$, so Y is a closed subtree. Then $\Gamma \setminus Y^o$ is a closed subtree of Γ. Moreover $N_\varepsilon(\Gamma \setminus Y^o) = N_\varepsilon(\Gamma) - 2$, $w(Y) > \varepsilon$ by an argument similar to that of (i).

If b), define a_i by $\{a_i\} = X_{l-1} \cap Z_i$ and, for $i \neq j, a_{ij} = a_i \wedge a_j = \max\{u : u \preceq a_i \text{ and } u \preceq a_j\}$ in the ordering of Γ arising from taking x_1 as root. Then $a_{ij} \in X_{l-1}$. Define $\varrho_i := \#\{a_{jk}; a_{jk} \preceq a_i\}$. Without loss of generality we may suppose $\varrho_1 = \max_i \varrho_i$ and that $a_{jk} \preceq a_{12}$ for all $a_{jk} \preceq a_1$. Define $Y = \{x; (\exists u) a_{12} \prec u \preceq a_1 \text{ or } a_{12} \prec u \preceq a_2 \text{ and } u \preceq x\} \cup \{a_{12}\}$. Then Y is a closed subtree of Γ and $Y \subseteq Z_1 \cup Z_2 \cup X_{l-1}$. If $\Gamma' = \Gamma \setminus Y^o$, Γ' is a closed subtree of Γ and it follows by arguments similar to those in (ii) that, since $\mathcal{C}$ is a chain of maximal length in $\mathcal{F}$ and $\mathcal{F}$ is a covering of Γ, $w(Y) > \varepsilon$ and

$$N_\varepsilon(\Gamma) - 3 \leq N_\varepsilon(\Gamma') \leq N_\varepsilon(\Gamma) - 2. \tag{3. 2}$$

Lemma 3.9 *If $N_\varepsilon(\Gamma) < \infty$ then $M_\varepsilon(\Gamma) \geq \frac{1}{3} N_\varepsilon(\Gamma)$.*

Proof. The proof is by induction on $N_\varepsilon(\Gamma)$. If $N_\varepsilon(\Gamma) = 1, w(\Gamma) \leq \varepsilon$ and $M_\varepsilon(\Gamma) = 1$. If $N_\varepsilon(\Gamma) = 2, l = 2$ and $\Gamma = X_1 \cup X_2$ and $M_\varepsilon(\Gamma) \geq 1$. If $N_\varepsilon(\Gamma) > 2$ the arguments of i) ii) iii) above and induction prove the result. For by i) $l \neq 1$ and by ii) or iii) $N_\varepsilon(\Gamma \setminus Y^o) < N_\varepsilon(\Gamma)$ so we may suppose $M_\varepsilon(\Gamma \setminus Y^o) \geq 1/3 N_\varepsilon(\Gamma \setminus Y^o)$ and then there exists $\mathcal{G}' \in \mathcal{L}_\varepsilon(\Gamma \setminus Y^o)$ with $\#\mathcal{G}' = M_\varepsilon(\Gamma \setminus Y^o)$. But then if $\mathcal{G} = \mathcal{G}' \cup \{Y\}$, $\mathcal{G} \in \mathcal{L}_\varepsilon(\Gamma)$ since $\omega(Y) > \varepsilon$ and $M_\varepsilon(\Gamma) \geq \#\mathcal{G} = M_\varepsilon(\Gamma \setminus Y^o) + 1 \geq \frac{1}{3} N_\varepsilon(\Gamma \setminus Y^o) + 1 \geq \frac{1}{3} N_\varepsilon(\Gamma)$. $\square$

Lemma 3.10 *Let K be a compact tree, let w be a function satisfying (3. 1) and, for every $(c,d) \in E(K)$, suppose that $w(c,.)$ is a continuous function on (c,d). Then there exists a set $\mathcal{G}$ of non-overlapping subtrees of K with $\omega(X) \geq \varepsilon$ for $X \in \mathcal{G}$ and*

$$\#\mathcal{G} \geq N_\varepsilon(K) - 3\#E(K). \tag{3. 3}$$

Proof. Let $\mathcal{F} \in S_\varepsilon(K)$ and $\#\mathcal{F} = N_\varepsilon(K)$. For $(c,d) \in E(K)$ define $\mathcal{K} := \{X : X \in \mathcal{F}, |X \cap (c,d)| > 0\}$ to be such that $\#\mathcal{K} > 3$. Set $\mathcal{K} = \{X_1, \ldots, X_n\}$, where $n = \#\mathcal{K}$ and $X_1 \preceq X_2 \ldots \preceq X_n$, where $X \preceq Y$ means that $x \preceq y$ for all $x \in X, y \in Y$. Then $X_i = (a_i, b_i) \subset (c,d)$ for $1 < i < n$. By the continuity of ω on (c,d) and the minimality of $\#\mathcal{F}$, there exists $b_2' \in (b_2, b_3)$ such that $\omega(a_2, b_2') = \varepsilon$. It follows that there are non-overlapping sub-intervals $X_2', \ldots, X_k'$ of (c,d), where $k = n-2$, for which $\omega(X_j') = \varepsilon$. The lemma follows from this.$\square$

Definition 3.11 *Let K be a subtree Γ. We denote by $\mathcal{P}(K)$ the set of partitions $\{\Gamma_i : i = 1, \ldots, n\}$ of K (i.e $\cup_{i=1}^n \Gamma_i = K$) by subtrees Γ_i of K such that $|\Gamma_i \cap \Gamma_j| = 0$ for $i \neq j$. For a given $\varepsilon > 0$ we define*

$$N(K,\varepsilon) \equiv N(K,\varepsilon,u,v) := \min\{n : \exists \quad \{\Gamma_i : i = 1, \ldots, n\} \in \mathcal{P}(K) \text{ and } A(\Gamma_i, u, v) \leq \varepsilon\};$$

$$M(K,\varepsilon) \equiv M(K,\varepsilon,u,v) := \max\{m : \exists \text{ non-overlapping subtrees } \Gamma_i \subseteq K, \; i = 1, \ldots, m, \text{ such that } A(\Gamma_i, u, v) > \varepsilon\}.$$

Note that, with $\omega(\cdot) = A(\cdot)$, $M_\varepsilon(\Gamma) \le M(\Gamma, \varepsilon) + 1$.

Hereafter we shall write T, T_K for $T_a, T_{a,K}$ respectively, unless there is a possibility of confusion. The following lemma will yield a one-dimensional approximation to T on a subtree K of Γ. We recall the notation

$$\mu(K) = \int_K d\mu, \quad d\mu = v^p dx.$$

Lemma 3.12 *Let K be a subtree of Γ and $v \in L^p(K)$, $1 \le p \le \infty$, with $\mu(K) \neq 0$. Then there exists $w_K \in \{L^p(K; d\mu)\}^*$ (in case $p = \infty$, w_K is also from $\{L^\infty(K)\}^*$) such that:*

$$w_K(1) = 1,$$

$$\|w_K\|_{\{L^p(K,d\mu)\}^*} = \frac{1}{\|v\|_{p,K}}, (\|w_K\|_{\{L^\infty(K)\}^*} = 1 \textit{ for } p = \infty)$$

and

$$\inf_{c\in\mathbf{C}} \|(\varphi - c)v\|_{p,K} \le \|(\varphi - w_K(\varphi))v\|_{p,K} \le 2 \inf_{c\in\mathbf{C}} \|(\varphi - c)v\|_{p,K} \tag{3. 4}$$

for all $\varphi \in L^p(K, d\mu)$. In the case $p = 2$

$$\inf_{c\in\mathbf{C}} \|(\varphi - c)v\|_{L^p(K)} = \|(\varphi - w_K(\varphi))v\|_{L^p(K)} \tag{3. 5}$$

where

$$w_K(\varphi) = \frac{1}{\mu(K)} \int_K \varphi d\mu.$$

Proof. Define the linear function w on the constants in $L^p(K, d\mu)$ with $w(c.1) = c$. Then $w(1) = 1$ and $\|w\|^p = 1/\mu(K)$ for $1 \le p < \infty$ while $\|w\| = 1$ when $p = \infty$. The existence of w_K follows by the Hahn-Banach theorem, and (3.4) is immediate. The case $p = 2$ is obvious. □

Remark 3.13 *For $1 \le p < \infty$ and $\Gamma = (0, \infty)$ the choice $w_K(\varphi) = \frac{1}{\mu(K)} \int_K \varphi d\mu$ is appropriate, and was that used in [2]. In [6] Lemma 2.4, when $p = \infty$ and $\Gamma = (a, b)$, w_K was defined as the limit along a filter base of subsets w_β of the unit ball in $\{L^\infty(K)\}^*$ defined by*

$$w_\gamma(\varphi) := \frac{1}{|A_\gamma|} \int_{A_\gamma} \varphi(x) dx, \qquad \varphi \in L^\infty(K).$$

Lemma 3.14 *Let K be a subtree of $\Gamma, a \in K, 1 < p \le \infty$, and suppose that $T_{a,K}$ is compact. Then*

$$A(K) = \|T_{a,K} - v\varpi | L^p(K) \to L^p(K)\|,$$

where ϖ is the bounded linear functional

$$\varpi(f) = \int_a^b fu$$

and $b \in K$ is such that $A(K) = \|T_{b,K}|L^p(K) \to L^p(K)\|$.

Proof. We know from Theorem 3.8 that there exists a $b \in K$ such that

$$A(K,u,v) = \|T_{b,K}|L^p(K) \to L^p(K)\| = \|T_{b,K}U|L^p(K) \to L^p(K)\|,$$

where U is the linear isometry defined in the proof of Lemma 3.2, and with respect to which

$$T_{b,K}f = v\chi_K \int_b^a fu\chi_K + T_{a,K}Uf.$$

Thus, if we define

$$Pf(x) = v(x)\chi_K(x) \int_a^b Ufu\chi_K,$$

we have

$$\|T_{a,K} - P|L^p(K) \to L^p(K)\| = A(K,u,v),$$

and the lemma follows.□

Lemma 3.15 *Let $\varepsilon > 0$, $1 \le p \le \infty$ and let K be a subtree of Γ. If $N := N(K,\varepsilon,u,v) < \infty$ then*

$$a_{N+1}(T_K) \le \gamma_p \varepsilon,$$

where $\gamma_p = 2$ when $p \ne 2$ and $\gamma_2 = 1$.

Proof. Let $\{\Gamma_i\}_1^N$ be the partition of K which defines $N := N(K,\varepsilon,v,u)$ in Definition 3.11 and set $Pf = \sum_{i=1}^N P_i f$ where

$$P_i f(x) := \chi_{\Gamma_i}(x)v(x)\left[\int_a^{a_i} uf + w_i\left(\int_{a_i}^x uf\chi_{\Gamma_i}\right)\right],$$

$T_K \equiv T_{a,K}, a_i$ is the point in Γ_i nearest a and $w_i \equiv w_{\Gamma_i}$ is the linear function from Lemma 3.12. Then $\operatorname{rank} P \le N$ and, on using Lemma 3.12, and setting $T_i \equiv T_{\Gamma_i}$,

$$\begin{aligned}
\|(T_K - P)f\|_{p,K}^p &= \sum_{i=1}^N \|(T_K - Pf)\|_{p,\Gamma_i}^p \\
&= \sum_{i=1}^N \|T_i - w_i\left(\int_{a_i}^{\cdot} uf\chi_{\Gamma_i} dt\right) v(\cdot)\|_{p,\Gamma_i}^p \\
&\le \gamma_p^p \left(\max_{i=1,\ldots,N} A(\Gamma_i,u,v)\right)^p \|f\|_{p,K}^p \\
&\le (\gamma_p\varepsilon)^p \|f\|_{p,K}^p,
\end{aligned}$$

whence the lemma.□

Lemma 3.16 *Let* $1 < p \leq \infty, \varepsilon > 0$, *and suppose that* T_K *is compact. Then, with* $N = N(K, \varepsilon, u, v) < \infty$,

$$a_{N+1}(T_K) \leq \varepsilon.$$

Proof. The same argument as in Lemma 3.15 applies but with

$$P_i f(x) = \chi_{\Gamma_i}(x) v(x) \left[\int_a^{a_i} uf + \varpi_i \left(\int_{a_i}^{b_i} uf\chi_{\Gamma_i} \right) \right],$$

where ϖ_i, b_i are as in Lemma 3.14 corresponding to $K = \Gamma_i$.□

Lemma 3.17 *Let* $\varepsilon > 0$, $1 \leq p \leq \infty$ *and let* K *be a subtree of* Γ. *Let* $\{\Gamma_i; i = 1, \ldots, M\}$ *be a set of non-overlapping subtrees of* K *such that* $A(\Gamma_i) \geq \varepsilon$ *for all* $1 \leq i \leq M$. *Then*

$$a_M(T_K) \geq \varepsilon.$$

Proof. Let $\lambda \in (0, 1)$. From the definition of $A(\Gamma_i)$ we know that for $i = 1, \ldots, M$ there exist $\phi_i \in L^p(\Gamma), \|\phi_i\|_p = 1$, with support in Γ_i such that

$$\inf_{\alpha \in \mathbf{C}} \|T_K \phi_i - \alpha v\|_{p,\Gamma_i} > \lambda A(\Gamma_i) \geq \lambda \varepsilon.$$

Let $P : L^p(K) \to L^p(K)$ be bounded with $\text{rank}(P) < M$. Then, there are constants $\lambda_1, \ldots, \lambda_M$ not all zero, such that

$$P\phi = 0, \qquad \phi := \sum_{i=1}^{M} \lambda_i \phi_i.$$

Then, noting that the following summation is over $\lambda_i \neq 0$, and denoting by a_i the point of Γ_i nearest a, where $T_K = T_{a,K}$,

$$\begin{aligned}
a_M^p \|\phi\|_{p,K}^p &\geq \|T_K\phi - P\phi\|_{p,K}^p = \|T_K\phi\|_{p,K}^p = \sum_{i=1}^{M} \|(T_K\phi)\chi_{\Gamma_i}\|_{p,K}^p \\
&= \sum_{i=1}^{M} \|\chi_{\Gamma_i}(x) v(x) \left(\int_{a_i}^{x} \lambda_i \phi_i(t) u(t) \chi_{\Gamma_i}(t) dt \right. \\
&\qquad \left. + \int_a^{a_i} \phi(t) u(t) dt \right) \|_{p,K}^p \\
&= \sum_{i=1}^{M} \| \left(T_K \phi_i(x) + v(x) \frac{\eta_i}{\lambda_i} \right) \lambda_i \|_{p,\Gamma_i}^p, \\
&\quad \text{where } \eta_i := \int_a^{a_i} \phi(t) u(t) dt,
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{i=1}^{M} \inf_{\alpha \in \mathbf{C}} \|(T_K \phi_i(x) - v(x)\alpha)\|_{p,\Gamma_i}^p |\lambda_i|^p \\
&\geq (\lambda\varepsilon)^p \sum_{i=1}^{M} \|\phi_i\|_{p,\Gamma_i}^p |\lambda_i|^p \geq (\lambda\varepsilon)^p \|\phi\|_{p,K}^p.
\end{aligned}
$$

□

Theorem 3.18 *Let $1 \leq p \leq \infty, \varepsilon > 0$ and $N = N(\Gamma, \varepsilon, u, v) < \infty$(see Definition 3.11). Then*

$$a_{N+1}(T) \leq \gamma_p \varepsilon$$

where $\gamma_p = 2$ when $p \neq 2$ and $\gamma_2 = 1$, and

$$a_{[\frac{N}{3}]-1}(T) > \varepsilon.$$

The measure of non-compactness of $T, \beta(T)$, satisfies

$$\beta(T) := \lim_{n\to\infty} a_n(T) \asymp \inf\{\varepsilon : N(\Gamma, \varepsilon, u, v) < \infty\},$$

where the symbol $\asymp$ means that the quotient of the two sides lies between positive constants. Hence, T is compact if and only if $N(\Gamma, \varepsilon, u, v) < \infty$ for all $\varepsilon > 0$. If T is compact, $\gamma_p = 1$ for $1 < p \leq \infty$.

Proof. The first inequality follows from Lemma 3.15. The second inequality follows from Lemmas 3.9 and 3.17 on putting $w(\cdot) := A(\cdot)$. The two inequalities imply the result about the measure of non-compactness, and hence the compactness, of T. The last statement is a consequence of Lemma 3.16.□

From Lemmas 3.10, 3.16, 3.17 and 3.18 with $w(X) := A(X)$ we derive

Lemma 3.19 *Let $1 < p < \infty$, $\varepsilon > 0$, and let K be a compact tree. Then*

$$a_{N+1}(T_K) \leq \varepsilon$$

and

$$a_{M-1}(T_K) > \varepsilon,$$

where $N = N(K, \varepsilon, u, v)$, and $M \geq N - 3\#E(K)$.

Since, by Lemma 3.2, $A(\Gamma_a)$ is independent of $a \in \Gamma$, the approximation numbers are independent of $a \in \Gamma$. Note that the above proof of Lemma 3.19 requires $A(c, .)$ to be continuous (see Lemma 3.10). This is not true for $p = 1$ or ∞.

4 Local properties of A

In this section we establish properties of the function A in Definition 3.1 which will be needed for the local asymptotic results in the next section.

Lemma 4.1 *Let u, v be constants over a real interval $I = (a_1, b_1)$ and $1 \le p \le \infty$. Then $A(I, u, v) = |v||u||I|\alpha_p$, where $\alpha_p = A((0,1), 1, 1)$.*

Proof. We have

$$\begin{aligned} A(I,u,v) &= \sup_{\|f\|_{p,I}\le 1} \inf_{\alpha\in\mathbf{C}} \|v\left(\int_{a_1}^{x} uf(t)dt - \alpha\right)\|_{p,I} \\ &= |v||u| \sup_{\|f\|_{p,I}\le 1} \inf_{\alpha\in\mathbf{C}} \|\int_{a_1}^{x} f(t)dt - \alpha\|_{p,I} \\ &= |v||u||I| \sup_{\|f\|_{p,(0,1)}\le 1} \inf_{\alpha\in\mathbf{C}} \|\int_{0}^{x} f(t)dt - \alpha\|_{p,(0,1)} \\ &= |v||u||I|A((0,1),1,1). \end{aligned}$$

□

Note that $\alpha_1 = \alpha_\infty = 1/2$ and $\alpha_2 = 1/\pi$.

Lemma 4.2 *Let $1 \le p \le \infty$, $u_1, u_2 \in L^{p'}(\Gamma)$ and $v \in L^p(\Gamma)$. Then*

$$|A(\Gamma, u_1, v) - A(\Gamma, u_2, v)| \le \|v\|_{p,\Gamma}\|u_1 - u_2\|_{p',\Gamma}.$$

Proof. We have

$$\begin{aligned} A(\Gamma,u_1,v) &= \sup_{\|f\|_p=1} \inf_{\alpha\in\mathbf{C}} \|v(x)\left(\int_{a}^{x} (u_1(t)-u_2(t)+u_2(t))f(t)dt - \alpha\right)\|_p \\ &\le \sup_{\|f\|_p=1} \inf_{\alpha\in\mathbf{C}} \Big[\|v(x)\int_{a}^{x} (u_1(t)-u_2(t))f(t)dt\|_{p,\Gamma} \\ &+ \qquad \|v(x)\left(\int_{a}^{x} u_2(t)f(t)dt - \alpha\right)\|_p\Big] \\ &\le \sup_{\|f\|_p=1} \inf_{\alpha\in\mathbf{C}} \left[\|v\|_p\|u_1-u_2\|_{p'} + \|v(x)\left(\int_{a_1}^{x} u_2(t)f(t)dt - \alpha\right)\|_p\right] \\ &\le \|v\|_p\|u_1-u_2\|_{p'} + A(\Gamma,u_2,v). \end{aligned}$$

The same holds with u_1, u_2 interchanged. □

Lemma 4.3 *Let $1 \le p \le \infty$, $u \in L^{p'}(\Gamma)$, and $v_1, v_2 \in L^p(\Gamma)$. Then*

$$|A(\Gamma, u, v_1) - A(\Gamma, u, v_2)| \le 2\|v_1 - v_2\|_p\|u\|_{p'}.$$

Proof. We have

$$\begin{aligned} A(\Gamma,u,v_1) &= \sup_{\|f\|_p=1} \inf_{\alpha\in\mathbf{C}} \|v_1(x)\left[\int_a^x u(t)f(t)dt-\alpha\right]\|_p \\ &= \sup_{\|f\|_p=1} \inf_{|\alpha|\le\|u\|_{p'}\|f\|_p} \|v_1(x)\left[\int_a^x u(t)f(t)dt-\alpha\right]\|_p. \end{aligned}$$

Since

$$\|v_1(x)[\int_a^x u(t)f(t)dt-\alpha]\|_p \le \|v_1-v_2\|_p\|u\|_{p'}\|f\|_p+\|(v_1-v_2)\alpha\|_p+\|v_2[\int_a^x u(t)f(t)dt-\alpha]\|_p$$

it follows that

$$\begin{aligned} A(\Gamma,u,v_1) &\le 2\|v_1-v_2\|_p\|u\|_{p'} + \sup_{\|f\|_p=1} \inf_{|\alpha|\le\|u\|_{p'}} \|v_2(x)\left[\int_a^x u(t)f(t)dt-\alpha\right]\|_p \\ &= 2\|v_1-v_2\|_p\|u\|_{p'} + A(\Gamma,u,v_2). \end{aligned}$$

Similarly with v_1 and v_2 interchanged. □

Lemma 4.4 *Let $1<p<\infty$ and suppose that $K_1, K_2, K_1 \subset K_2$, are compact subtrees of Γ. Then*

$$|A(K_1)-A(K_2)| \to 0 \qquad as\ |K_2 \setminus K_1| \to 0$$

Proof. We see that $A(K_1) = A(\Gamma, u\chi_{K_1}, v\chi_{K_1})$ and $A(K_2) = A(\Gamma, u\chi_{K_2}, v\chi_{K_2})$. The lemma then follows from Lemmas 4.2 and 4.3.□

In order to treat the cases $p = 1,\infty$ we need the following terminology.

Definition 4.5 *Let $u \in L^\infty(\Gamma)$. Then*

$$u_s := \lim_{\varepsilon\to 0_+} \|u\chi_{B(t,\varepsilon)}\|_{\infty,\Gamma}$$

where $B(t,\varepsilon)$ is the ball center t, radius ε on Γ.

Definition 4.6 *Let g be a function defined on a real interval I. Then*

$$g^*(x) := \inf\{t; g_*(t) \ge x\},$$

where $g_*(t) := |\{x \in I; g(x) \ge t\}|$. The function g^* is the non–increasing rearrangement of g.

Note that since we have $\ge$ in the definitions above, g_* and g^* are left–continuous functions. For the case $p=\infty$ we have the following two lemmas.

Lemma 4.7 *Let I be a bounded interval, $\gamma, \delta \in \mathbf{R}$ with $\delta \geq v_s(t) \geq 0$ on I, and let $p = \infty$. Then*

$$A(I; \gamma, \delta) \geq A(I; \gamma, v_s) \geq 1/2|\gamma| \|(v_s \chi_I)^*(t)t\|_{\infty,(0,|I|)}.$$

Proof. See [6; Lemma 4.5]. □

Lemma 4.8 *Let I be a bounded interval, $\gamma, \delta \in \mathbf{R}$ with $\delta \geq v_s(t) \geq 0$ on I and let $p = \infty$. Then for any $\alpha > 1$*

$$A(I; \gamma, \delta) - A(I; \gamma, v_s) \leq \frac{\alpha}{2} \int |\gamma|(\delta - v_s(t))dt + \frac{|\gamma|\delta|I|}{2\alpha}.$$

Proof. See [6, Lemma 4.6]. □

For the case $p = 1$ we have

Lemma 4.9 *Let I be a bounded interval, $\gamma, \delta \in \mathbf{R}$ with $\delta \geq u_s(t) \geq 0$ and $\infty > u_1(t) \geq u_2(t) \geq 0$ on I. Then for $p = 1$*

$$A(I; u_1, \gamma) \geq A(I; u_2, \gamma)$$

and

$$A(I; u_s, \gamma) \geq 1/4|\gamma| \|(u_s \chi_I)^*(t)t\|_{\infty,(0,|I|)}.$$

Proof. In the first inequality

$$A(I; u_1, \gamma) = |\gamma| \sup_{\|f\|_{1,I} \leq 1} \inf_{\alpha \in \mathbf{C}} \frac{\| \int_a^x u_1 f - \alpha\|_{1,I}}{\|f\|_{1,I}}.$$

For any $\|f_2\|_{1,I} \leq 1$ there exists f_1 such that $\|f_1\|_{1,I} \leq \|f_2\|_{1,I} \leq 1$ and $\int_a^x u_1 f_1 = \int_a^x u_2 f_2$. (Put $f_1(t) := f_2(t)u_2(t)/u_1(t)$ if $u_1(t) \neq 0$ and $f_1(t) := f_2(t)$ otherwise.) Then $A(I; u_1, \gamma) \geq A(I; u_2, \gamma) \geq 0$.

In the second inequality, we have

$$\begin{aligned} A(I; u_s, \gamma) &= |\gamma| \sup_{\|f\|_{1,I}=1} \inf_{\alpha \in \mathbf{C}} \| \int_a^x u_s f - \alpha\|_{1,I} \\ &\geq |\gamma| \sup_{\|f\|_{1,I}=1} \inf_{\alpha \in \mathbf{C}} \| \int_a^x \chi_{M_\beta} \beta f - \alpha\|_{1,I} \end{aligned}$$

where $M_\beta := \{y \in I; u_s(y) \geq \beta\}$ and $0 \leq \beta \leq \|u_s\|_{\infty,I}$. Put $f = \delta_{x_\beta}$, where $x_\beta \in I = (a, b)$ and $|M_\beta \cap (a, x_\beta)| = |M_\beta \cap (x_\beta, b)| = 1/2|M_\beta|$. Then

$$\begin{aligned} A(I; u_s, \gamma) &\geq |\gamma| \inf_{\alpha \in \mathbf{C}} \|\chi_{(x_\beta,b)}\beta - \alpha\|_{1,I} \\ &= |\gamma| \inf_{\beta > \alpha > 0} (\alpha(x_\beta - a), (\beta - \alpha)(b - x_\beta)) \end{aligned}$$

$$
\begin{aligned}
&= |\gamma||\beta| \inf_{1>\alpha>0} (\alpha(x_\beta - a), (1-\alpha)(b - x_\beta)) \\
&= |\gamma||\beta| \frac{b - x_\beta}{b-a}(x_\beta - a) \\
&\geq |\gamma||\beta|| \frac{M_\beta}{2}| \frac{b - a - |M_\beta/2|}{b-a} \\
&\geq |\gamma||\beta||M_\beta| \frac{1}{2} \frac{b - a - (\frac{b-a}{2})}{b-a} \\
&= \frac{1}{4}|\gamma||\beta||M_\beta|.
\end{aligned}
$$

Hence, we have for every $0 \leq \beta \leq \|u_s\|_{\infty,I}$

$$
A(I; u_s, \gamma) \geq |\gamma||\beta||M_\beta| \frac{1}{4}
$$

and so

$$
A(I; u_s, \gamma) \geq |\gamma| \frac{1}{4} \|t(u_s \chi_I)^*(t)\|_{\infty,(0,|I|)}.
$$

□

Lemma 4.10 *Let I be a bounded interval, $\gamma, \delta \in \mathbf{R}$ with $\delta \geq u_s(t) \geq 0$ on I and $p = 1$. Then for any $\alpha > 1$*

$$
A(I; \delta, \gamma) - A(I; u_s, \gamma) \leq \frac{\alpha}{2} \int_I |\gamma|(\delta - u_s(t))dt + \frac{|\gamma|\delta|I|}{2\alpha}.
$$

Proof. From Lemmas 4.9 and 4.1 we have

$$
0 \leq A(I; \delta, \gamma) - A(I; u_s, \gamma) \leq \frac{1}{2}|\gamma||\delta||I| - \frac{1}{4}|\gamma|\|(u_s \chi_I)^*(t)t\|_\infty.
$$

The rest of the proof is similar to that in [6, Lemma 4.6] on using Lemma 4.8 instead of [6, Lemma 4.5]. □

5 Local asymptotic results

5.1 Case $1 < p < \infty$

Lemma 5.1 *Let K be a compact subtree of Γ and $1 < p < \infty$. Then*

$$
\alpha_p \int_K |u||v| = \lim_{\varepsilon \to 0_+} \varepsilon N(K, \varepsilon, u, v),
$$

$$
\alpha_p \int_K |u||v| = \lim_{\varepsilon \to 0_+} \varepsilon M(K, \varepsilon, u, v),
$$

where $N(K, \varepsilon, u, v)$ and $M(K, \varepsilon, u, v)$ are defined in Definition 3.11, and $\alpha_p = A((0,1), 1, 1)$ (see Lemma 4.1).

Proof. Since K is a compact tree it has a bounded number of vertices, i.e. K is a finite union of intervals. The argument in [2, Theorem 5], with A replacing the function L there, continues to go through and yields the first equation of the lemma. The second identity follows from the first identity and Lemma 3.10 since A is a continuous function on an interval. □

Lemma 5.2 *Let $1 < p < \infty$. Then*

$$\alpha_p \int_\Gamma |u||v| \leq \liminf_{\varepsilon \to 0+} \varepsilon N(\Gamma, \varepsilon, u, v).$$

Proof. There exist compact subtrees Γ_n of Γ, $n = 1, 2, \ldots$ such that $\Gamma_n \subset \Gamma$ and $\Gamma_n \to \Gamma$ (i.e. $|\Gamma - \Gamma_n| \to 0$) as $n \to \infty$. By Lemma 5.1 we have

$$\alpha_p \int_{\Gamma_n} |u||v| = \lim_{\varepsilon \to 0_+} \varepsilon N(\Gamma_n, \varepsilon, u, v) \leq \liminf_{\varepsilon \to 0_+} \varepsilon N(\Gamma, \varepsilon, u, v),$$

whence the result. □

Theorem 5.3 *Let $1 < p < \infty$, $u \in L^{p'}(\Gamma)$ and $v \in L^p(\Gamma)$. Then*

$$\lim_{\varepsilon \to 0_+} \varepsilon N(\Gamma, \varepsilon, u, v) = \alpha_p \int_\Gamma |u||v|,$$

$$\lim_{\varepsilon \to 0_+} \varepsilon M(\Gamma, \varepsilon, u, v) = \alpha_p \int_\Gamma |u||v|.$$

Proof. Let $\mathcal{L}$ be a maximal such that $\#\mathcal{L} = M(\Gamma, \varepsilon) \equiv M(\Gamma, \varepsilon, u, v)$ and $\mathcal{S}$ a minimal cover such that $\#\mathcal{S} = N(\Gamma, \varepsilon) \equiv N(\Gamma, \varepsilon, u, v)$.

Given $\eta > 0$, choose a compact subtree $K \subset \Gamma$ such that $\|u\chi_{\Gamma \setminus K}\|_{p'} \leq \eta$ and $\|v\chi_{\Gamma \setminus K}\|_p \leq \eta$.

Set

$$\begin{aligned}
\mathcal{L}_1 &= \{\Gamma' : \Gamma' \in \mathcal{L}, \Gamma' \subseteq K\}, \\
\mathcal{L}_2 &= \{\Gamma' : \Gamma' \in \mathcal{L}, \Gamma' \subseteq \Gamma \setminus K\}, \\
\mathcal{L}_3 &= \mathcal{L} \setminus \{\mathcal{L}_1 \cup \mathcal{L}_2\},
\end{aligned}$$

and similarly for $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ with respect to $\mathcal{S}$.

We know that $\Gamma \setminus K$ is the union of disjoint connected components $\{\Gamma_i^*\}$ and $\#\{\Gamma_i^*\} \leq \#\partial K$. Also, if $X \in \mathcal{L}_2 \cup \mathcal{S}_2$ then $X \subseteq \Gamma_i^*$ for some i. Thus $\#\mathcal{L}_2 \leq \sum_i M(\Gamma_i^*, \varepsilon)$.

Consider now the union of $\mathcal{S}_1, \mathcal{S}_3$ and those subtrees in the definition of the $N(\Gamma_i^*, \varepsilon)$. This covers Γ and so

$$\#\mathcal{S}_2 \leq \sum_i N(\Gamma_i^*, \varepsilon) \leq 3 \sum_i M(\Gamma_i^*, \varepsilon) + 3\#\partial K \tag{5. 1}$$

by Lemma 3.9. Let $\Gamma_i^*(j)$ be the subtrees in the definition of $M(\Gamma_i^*, \varepsilon)$. Then

$$\varepsilon \# \mathcal{L}_2 \le \sum_{i,j} A(\Gamma_i^*(j)) \le \sum_{i,j} \|u\|_{p', \Gamma_i^*(j)} \|v\|_{p, \Gamma_i^*(j)} \le \|u\chi_{\Gamma \setminus K}\|_{p'} \|v\chi_{\Gamma \setminus K}\|_p \le \eta^2.$$

Since $\# \mathcal{L}_1 \le M(K, \varepsilon)$

$$M(\Gamma, \varepsilon) \le M(K, \varepsilon) + \# \mathcal{L}_2 + \# \mathcal{L}_3,$$

$$0 < \varepsilon[M(\Gamma, \varepsilon) - M(K, \varepsilon)] \le \varepsilon(\# \mathcal{L}_2 + \# \mathcal{L}_3) \le \eta^2 + \varepsilon \# \partial K.$$

Then, by Lemma 5.1,

$$0 \le \limsup_{\varepsilon \to 0} \varepsilon M(\Gamma, \varepsilon) - \alpha_p \int_K |uv| \le \eta^2.$$

Now let $K \to \Gamma$ ($\eta \to 0$) to get

$$\limsup_{\varepsilon \to 0} \varepsilon M(\Gamma, \varepsilon) = \alpha_p \int_\Gamma |uv|.$$

We get the same for $\liminf$ and so

$$\lim_{\varepsilon \to 0} \varepsilon M(\Gamma, \varepsilon) = \alpha_p \int_\Gamma |uv|.$$

Since $N(K, \varepsilon) + \# \mathcal{S}_2 + \# \mathcal{S}_3 \ge N(\Gamma, \varepsilon)$,

$$0 \le N(\Gamma, \varepsilon) - N(K, \varepsilon) \le \# \mathcal{S}_2 + \# \mathcal{S}_3 \le 3 \sum_i M(\Gamma_i^*(j), \varepsilon) + 3\# \partial K + \# \partial K$$

by (5.1). Hence, as before,

$$\lim_{\varepsilon \to 0} \varepsilon N(\Gamma, \varepsilon) = \alpha_p \int_\Gamma |uv|.$$

□

Corollary 5.4 *Let* $1 < p < \infty$, $u \in L^{p'}(\Gamma)$ *and* $v \in L^p(\Gamma)$. *Then*

$$\lim_{n \to \infty} n a_n(T) = \alpha_p \int_\Gamma |u||v|.$$

Proof. Note that the application of Theorem 3.18 to Theorem 5.3 implies that $\lim_{n \to \infty} a_n(T) = 0$ and hence that T is compact.

Let $\{\Gamma_l\}_{l=1}^\infty$ be as in the proof of Lemma 5.2, and set $T_l = T_{a, \Gamma_l}$ for some $a \in \Gamma$. Since Γ_l is compact then from Lemma 3.19 and Theorem 5.1 we have

$$\lim_{n \to \infty} n a_n(T_l) = \alpha_p \int_{\Gamma_l} |u||v|.$$

An operator of rank $< n$ on Γ_l can be considered as the restriction to Γ_l of such an operator on Γ and also $\|T|L^p(\Gamma) \to L^p(\Gamma)\| \geq \|T_l|L^p(\Gamma_l) \to L^p(\Gamma_l)\|$ if $T = T_a$ and $a \in \Gamma_l$. It follows that $a_n(T) \geq a_n(T_l)$ and so

$$\liminf_{n\to\infty} na_n(T) \geq \alpha_p \int_{\Gamma_l} |u||v|.$$

But, we know from Lemma 5.3 that

$$\lim_{\varepsilon\to 0_+} \varepsilon N(\Gamma, \varepsilon) = \alpha_p \int_{\Gamma} |u||v|.$$

and so, by Lemma 3.16

$$\limsup_{n\to\infty} na_n(T) \leq \alpha_p \int_{\Gamma} |u||v|.$$

Hence the corollary is proved.□

5.2 The cases $p = \infty$ and $p = 1$.

The analogies of Lemma 5.1 for $p = \infty$ and $p = 1$ are, respectively

Lemma 5.5 *Let K be a compact subtree of Γ, and $p = \infty$. Then*

$$\frac{1}{2}\int_K |u||v_s| \leq \liminf_{\varepsilon\to 0_+} \varepsilon N(K, \varepsilon, u, v) \leq \limsup_{\varepsilon\to 0_+} \varepsilon N(K, \varepsilon, u, v) \leq \frac{3}{2}\int_K |u||v_s|$$

where v_s is defined in Definition 5.4.

Lemma 5.6 *Let K be a compact subtree of Γ and $p = 1$. Then*

$$\frac{1}{2}\int_K |u_s||v| \leq \liminf_{\varepsilon\to 0_+} \varepsilon N(K, \varepsilon, u, v) \leq \limsup_{\varepsilon\to 0_+} \varepsilon N(K, \varepsilon, u, v) \leq \frac{3}{2}\int_K |u_s||v|.$$

Both lemmas follow from the results for intervals in [6] since K is a finite union of intervals. Lemmas 5.5 and 5.6 yield, as in Lemma 5.2,

Lemma 5.7 *For $p = \infty$*

$$\frac{1}{2}\int_{\Gamma} |u||v_s| \leq \liminf_{\varepsilon\to 0_+} \varepsilon N(\Gamma, \varepsilon, u, v)$$

and for $p = 1$

$$\frac{1}{2}\int_{\Gamma} |u_s||v| \leq \liminf_{\varepsilon\to 0_+} \varepsilon N(\Gamma, \varepsilon, u, v).$$

Lemma 5.8 *Let $u \in L^{p'}(\Gamma)$ and $v \in L^p(\Gamma)$. Then for $p = \infty$*

$$\frac{1}{2}\int_\Gamma |u||v_s| \le \liminf_{\varepsilon\to 0_+} \varepsilon N(\Gamma,\varepsilon,u,v) \le \limsup_{\varepsilon\to 0_+} \varepsilon N(\Gamma,\varepsilon,u,v) \le \frac{3}{2}\int_\Gamma |u||v_s|$$

and for $p = 1$

$$\frac{1}{2}\int_\Gamma |u_s||v| \le \liminf_{\varepsilon\to 0_+} \varepsilon N(\Gamma,\varepsilon,u,v) \le \limsup_{\varepsilon\to 0_+} \varepsilon N(\Gamma,\varepsilon,u,v) \le \frac{3}{2}\int_\Gamma |u_s||v|.$$

Proof. Let $p = \infty$. We need only prove the last inequality. Let $\{\Gamma_l\}_{l=1}^\infty$ be compact subtrees of Γ which are such that

$$\left|\int_\Gamma |u||v_s| - \int_{\Gamma_l} |u||v_s|\right| \le \frac{1}{l}$$

and

$$\|u\|_{1,\Gamma\setminus\Gamma_l} \le \frac{1}{l}.$$

Fix $l \in \mathbf{N}$. There exist intervals $W(j)$ in Γ_l and step functions u_n, v_n on Γ_l,

$$u_n = \sum_{j=1}^m \xi_j \chi_{W(j)}, \qquad v_n = \sum_{j=1}^m \eta_j \chi_{W(j)},$$

which are such that

$$\|u - u_n\|_{1,\Gamma_l} < \frac{1}{n}, \quad \int_{\Gamma_l} |u(t)|(v_n(t) - v_s(t))dt < \frac{1}{n}$$

and $\|v_s\|_{\infty,\Gamma} \ge v_n(t) \ge v_s(t)$ on Γ_l; cf [6, Theorem 4.7].

Let $M := M(\Gamma,\varepsilon)$ and let $\{\Gamma_i^M\}_{i=1}^M$ be a maximal set of subtrees of Γ in the definition of M (see Definition 3.11). Then, because Γ_l is a compact subtree of Γ, we have $M - 2m - \#V(\Gamma_l) - \#\partial\Gamma_l \le \#\mathbf{K}$, where

$$\mathbf{K} := \{\Gamma_j : \Gamma_j \in \{\Gamma_k^M\}_{k=1}^M, \text{ and there exists } i \text{ such that } W(i) \supset \Gamma_j \text{ or } \Gamma_j \subset \Gamma\setminus\Gamma_l\}.$$

On using Lemmas 4.6 and 4.7, we have

$$\begin{aligned}
\varepsilon(M - 2m &- \#V(\Gamma_l) - \#\partial\Gamma_l) \le \sum_{k\in\mathbf{K}} A(\Gamma_k^M, u, v) \\
&\le \sum_{k\in\mathbf{K}} \big(A(\Gamma_k^M, u_n, v_n) + \big[A(\Gamma_k^M, u, v) - A(\Gamma_k^M, u_n, v)\big] \\
&\qquad + \big[A(\Gamma_k^M, u_n, v) - A(\Gamma_k^M, u_n, v_n)\big]\big) \\
&\le \frac{1}{2}\sum_{j=1}^m |\xi_j||\eta_j||W(j)|
\end{aligned}$$

$$
\begin{aligned}
& +\sum_{j=1}^{M}\left(\|u-u_n\|_{1,\Gamma_j^M}\|v\|_{\infty,\Gamma_J^M}\right) \\
& +\sum_{k\in\mathbf{K};\exists i,W(i)\supset\Gamma_k^M}\left[A(\Gamma_k^M,u_n,v)-A(\Gamma_k^M,u_n,v_n)\right] \\
& +\sum_{k\in\mathbf{K};\Gamma_k^M\subset\Gamma\setminus\Gamma_l}\left[A(\Gamma_k^M,u_n,v)-A(\Gamma_k^M,u_n,v_n)\right] \\
\leq\ & \frac{1}{2}\sum_{i=1}^{m}|\xi_j||\eta_j||W(j)|+\|u-u_n\|_1\|v\|_\infty \\
& +\sum_{k\in K;\exists i,W(i)\supset\Gamma_k^M}\left[\frac{\alpha}{2}\int_{\Gamma_k^M}(v_n-v_s)|\xi_i|dt+\frac{|\xi_i||\eta_i|}{2\alpha}|\Gamma_k^M|\right] \\
& +\sum_{k\in\mathbf{K},\Gamma_k^M\subset\Gamma\setminus\Gamma_l}\left[A(\Gamma_k^M,0,v)-A(\Gamma_k^M,0,0)\right] \\
\leq\ & \frac{1}{2}\sum_{i=1}^{m}|\xi_j||\eta_j||W(j)|+(\frac{1}{n}+\frac{1}{l})\|v_s\|_\infty \\
& +\frac{\alpha}{2}\int_\Gamma(v_n-v_s)|u_n|dt \\
& +\frac{1}{2\alpha}\int_\Gamma|u_n||v_n|dt \\
\leq\ & \frac{1}{2}\int_\Gamma|u||v_s|+c\left(\alpha\frac{1}{n}+\frac{1}{\alpha}+\frac{1}{n}+\frac{1}{l}\right)
\end{aligned}
$$

for some constant c independent on ε. We therefore conclude that

$$\frac{1}{3}\limsup_{\varepsilon\to 0_+}\varepsilon N(\Gamma,\varepsilon,u,v)\leq\frac{1}{2}\int_\Gamma|u||v_s|+K(\alpha\frac{1}{n}+\frac{1}{\alpha}),$$

whence

$$\frac{1}{3}\limsup_{\varepsilon\to 0_+}\varepsilon N(\Gamma,\varepsilon,u,v)\leq\frac{1}{2}\int_\Gamma|u||v_s|.$$

The case $p=1$ is similar. $\square$

From Lemmas 5.8 and 3.18 we derive

Lemma 5.9 *Let $u\in L^{p'}(\Gamma)$ and $v\in L^p(\Gamma)$. Then for $p=\infty$*

$$\frac{1}{6}\int_\Gamma|u||v_s|\leq\liminf_{n\to\infty}na_n(T)\leq\limsup_{n\to\infty}na_n(T)\leq 3\int_\Gamma|u||v_s|$$

and for $p=1$

$$\frac{1}{6}\int_\Gamma|u_s||v|\leq\liminf_{n\to\infty}na_n(T)\leq\limsup_{n\to\infty}na_n(T)\leq 3\int_\Gamma|u_s||v|.$$

6 The main results for $1 < p < \infty$.

We suppose throughout this section that $T := T_a$ and $\Gamma := \Gamma_a$ for some $a \in \Gamma$. Also we write $\|T_K\|$ for $\|T|L^p(K) \to L^p(K)\|$, for any $K \subseteq \Gamma$.

With $U(x) := \int_a^x |u(t)|^{p'} dt$ $(x \in \Gamma)$ we define Z_k to be the closure of

$$\{x : x \in \Gamma, 2^{\frac{kp'}{p}} \le U(x) < 2^{\frac{(k+1)p'}{p}}\}. \tag{6. 1}$$

Here k may be any integer if $u \in L^{p'}_{loc}(\Gamma) \setminus L^{p'}(\Gamma)$, while, if $u \in L^{p'}(\Gamma)$, $2^k \le \|u\|^p_{p',\Gamma}$; we refer to these values of k as the *admissible* values.

We have that $Z_k = \bigcup_{i=1}^{n_k} Z_{k,i}$, where the $Z_{k,i}$ are the connected components of Z_k. Corresponding to each admissible k we set

$$\sigma^p_{k,i} := 2^k \mu(Z_{k,i}) \text{ for } i \in \{1, \dots, n_k\} \tag{6. 2}$$

and

$$\sigma^p_k := 2^k \mu(Z_k). \tag{6. 3}$$

For non–admissible k we set $\sigma_k = 0$. We also set $\sigma_{k,i} = 0$ for $i \notin \{1, \dots, n_k\}$.

Let

$$B_{k,i} := \#\partial Z_{k,i} - 1; \tag{6. 4}$$

that is, $B_{k,i}$ is the number of boundary points of $Z_{k,i}$ excluding its root.

Lemma 6.1

$$\sup_{k \in \mathbf{Z}} \max_{1 \le j \le n_k} \sigma_{k,i} \le \|T\|. \tag{6. 5}$$

Proof. This follows from [7, Proposition 5.1], which asserts that

$$\sup_{x \in \Gamma} \|u\chi_{(a,x)}\|_{p'} \|v\chi_{(a,x)^c}\|_p \le \|T\|,$$

where $(a,x)^c = \{y \in \Gamma : x \preceq_a y\}$. For then, by (6. 1),

$$\begin{aligned} \|T\| &\ge \sup_{x \in \Gamma} U(x)^{1/p'} \left(\int_{y \succeq x} |v(y)|^p dy \right)^{1/p} \\ &\ge \sup_{k,i} 2^{k/p} \mu(Z_{k,i})^{1/p} \\ &= \sup_{k,i} \sigma_{k,i}. \end{aligned}$$

□

Lemma 6.2 *Let Γ' be a subtree of $\Gamma = \Gamma_a$ and $b = b(\Gamma')$ the nearest point of Γ' to a. Then, for any $c > 4$, there exist $X = X(\Gamma') \in \mathbf{I}_b(\Gamma')$ and $k' = k'(\Gamma') \in \mathbf{Z}$ such that, with $T' = T_{\Gamma'}$ and $Y = Y(\Gamma') = \Gamma' \setminus X$,*

$$\|T'\| \le 2^{2/p} c \{\sum_{i \in S} 2^{k'} \mu(Y \cap Z_{k',i})\}^{1/p}, \tag{6. 6}$$

where $S = S(\Gamma') = \{i : \mu(Y \cap Z_{k',i}) > 0\}$.

Proof. From Theorem 2.4, for $c > 4$, there exists $X \in \mathbf{I}_b(\Gamma')$ such that

$$\begin{aligned}
\|T'\| &\leq c\frac{\mu(Y)^{1/p}}{\alpha_X} \\
&\leq c \min_{t \in \partial X \setminus \{b\}} \left(\int_b^t |u|^{p'} dx \right)^{1/p'} \mu(Y)^{1/p} \\
&\leq c \min_{t \in \partial X \setminus \{b\}} [U(t) - U(b)]^{1/p'} [\sum_{i,k} \mu(Y \cap Z_{k,i})]^{1/p} \\
&\leq c \min_{t \in (\partial X \setminus \{b\}) \cap Z_{\gamma_o}} [U(t) - U(b)]^{1/p'} [\sum_{i,k} \mu(Y \cap Z_{k,i})]^{1/p},
\end{aligned}$$

where $\gamma_0 = \min\{k : \mu(Y \cap Z_k) > 0\}$. Since $\mu(\Gamma) < \infty$, we may assume that Y is compact and hence

$$\max_{k \geq \gamma_0} \sum_{i=1}^{n_k} 2^k \mu(Y \cap Z_{k,i})$$

is attained, and so

$$\begin{aligned}
\sum_{k,i} \mu(Y \cap Z_{k,i}) &= \sum_{k \geq \gamma_0} 2^{-k} \sum_{i=1}^{n_k} 2^k \mu(Y \cap Z_{k,i}) \\
&\leq 2^{1-\gamma_0} \sum_{i=1}^{n_{k'}} 2^{k'} \mu(Y \cap Z_{k',i})
\end{aligned}$$

say, for some $k' \geq \gamma_0$. Hence

$$\|T'\| \leq c[2^{(\gamma_0+1)p'/p} - 2^{\gamma_0 p'/p}]^{1/p'} [2^{1-\gamma_0} \sum_{i=1}^{n_{k'}} 2^{k'} \mu(Y \cap Z_{k',i})]^{1/p},$$

whence (6. 6). $\square$

Lemma 6.3 *Let $\{\Gamma_i\}_{\mathcal{L}}$ be a finite set of non-overlapping subtrees of Γ and set $T_l = T_{\Gamma_l}$. Then,*

$$\sum_{l \in \mathcal{L}} \|T_l\|^q \leq (2^{2/p+2})^q \sum_{(k,i) \in \eta} B_{k,i}^{q/p'} \sigma_{k,i}^q \qquad \textit{if } 1 \leq q \leq p \tag{6. 7}$$

and

$$\sum_{l \in \mathcal{L}} \|T_l\|^q \leq (2^{2/p+2})^q \sum_{k \in \eta_0} \sigma_k^q \qquad \textit{if } p \leq q < \infty, \tag{6. 8}$$

where η and η_0 are finite sets.

Proof. Let $\Gamma_\lambda \in \{\Gamma_l\}_{\mathcal{L}}$, and, in the notation of Lemma 6.2, set $b_l = b(\Gamma_l)$, $k_l = k'(\Gamma_l)$, $Y_l = Y(\Gamma_l)$ and $S_l = S(\Gamma_l)$. There are two cases to consider for Γ_λ:

(i) $b_\lambda \in Z_{k_\lambda}$. In this case $b_\lambda \in Z_{k_\lambda, i_\lambda}$, $S_\lambda = S(\Gamma_\lambda) = \{i_\lambda\}$ and, for any $c > 4$,

$$\|T_\lambda\| \leq (2^{2/p}c)\sigma_{k_\lambda, i_\lambda}.$$

(ii) $b_\lambda \notin Z_{k_\lambda}$. Denote by Λ the subset of $\mathcal{L}$ which is such that for $l \in \Lambda$, $b_l \in Z_{k_\lambda, i_l}$ for some unique $i_l \in S_\lambda$ and so $S_l = \{i_l\}$.

Set $\Lambda_i = \{l \in \Lambda : i_l = i\}$. Then, by (6. 6), for $q \geq 1$,

$$\begin{aligned}
\sum_{l \in \Lambda} \|T_l\|^q &= \sum_{i \in S_\lambda} \sum_{l \in \Lambda_i} \|T_l\|^q \\
&\leq (2^{2/p}c)^q \sum_{i \in S_\lambda} \sum_{l \in \Lambda_i} \left\{2^{k_\lambda} \mu(Y_l \cap Z_{k_\lambda, i})\right\}^{q/p} && (6.\ 9) \\
&\leq (2^{2/p}c)^q \sum_{i \in S_\lambda} \left\{ \sum_{l \in \Lambda_i} \left[2^{k_\lambda} \mu(Y_l \cap Z_{k_\lambda, i})\right]^{1/p} \right\}^q \\
&\leq (2^{2/p}c)^q \sum_{i \in S_\lambda} \left[\left\{ \sum_{l \in \Lambda_i} 2^{k_\lambda} \mu(Y_l \cap Z_{k_\lambda, i}) \right\}^{1/p} \left\{ \sum_{l \in \Lambda_i} 1 \right\}^{1/p'} \right]^q \\
&\leq (2^{2/p}c)^q \sum_{i \in S_\lambda} \left(B_{k_\lambda, i}^{1/p'} \sigma_{k_\lambda, i} \right)^q. && (6.\ 10)
\end{aligned}$$

Also in case (ii), from (6. 6),

$$\|T_\lambda\| \leq (2^{2/p}c)(\sum_{i \in S_\lambda} \sigma_{k_\lambda, i}^p)^{1/p}.$$

Hence, if $1 \leq q \leq p$,

$$\begin{aligned}
\|T_\lambda\|^q &\leq (2^{2/p}c)^q (\sum_{i \in S_\lambda} \sigma_{k_\lambda, i}^q) \\
&\leq (2^{2/p}c)^q \sum_{i \in S_\lambda} \left(B_{k_\lambda, i}^{1/p'} \sigma_{k_\lambda, i} \right)^q. && (6.\ 11)
\end{aligned}$$

If $q \geq p$, then from (6. 9),

$$\begin{aligned}
\sum_{l \in \Lambda} \|T_l\|^q &\leq (2^{2/p}c)^q \sum_{i \in S_\lambda} \left\{ \sum_{l \in \Lambda_i} 2^{k_\lambda} \mu(Y_l \cap Z_{k_\lambda, i}) \right\}^{q/p} \\
&\leq (2^{2/p}c)^q \sum_{i \in S_\lambda} \sigma_{k_\lambda, i}^q \\
&\leq (2^{2/p}c)^q \left(\sum_{i \in S_\lambda} \sigma_{k_\lambda, i}^p \right)^{q/p} \\
&\leq (2^{2/p}c)^q \sigma_{k_\lambda}^q. && (6.\ 12)
\end{aligned}$$

Also, by (6. 6),

$$\begin{aligned} \|T_\lambda\|^q &\leq (2^{2/p}c)^q \left\{ \sum_{l \in S_\lambda} 2^{k_\lambda} \mu(Y_\lambda \cap Z_{k_\lambda, i}) \right\}^{q/p} \\ &\leq (2^{2/p}c)^q \sigma_{k_\lambda}^q. \end{aligned} \tag{6. 13}$$

The lemma follows from (6. 10)-(6. 13) since $c > 4$ is arbitrary. □

Theorem 6.4 *For $1 < p < \infty$, let u, v satisfy (2. 1) and suppose that $B_{k,i}^{1/p'} \sigma_{k,i} \in l^1(\mathbf{Z} \times \mathbf{N})$. Then*

$$\lim_{\varepsilon \to 0} \varepsilon M(\Gamma, \varepsilon) = \alpha_p \int_\Gamma |u||v|, \tag{6. 14}$$

$$\lim_{\varepsilon \to 0} \varepsilon N(\Gamma, \varepsilon,) = \alpha_p \int_\Gamma |u||v|, \tag{6. 15}$$

and

$$\lim_{n \to \infty} n a_n(T) = \alpha_p \int_\Gamma |u||v|. \tag{6. 16}$$

Proof. Given $\eta > 0$, we choose l to be such that $\sum_{k \geq l} \sum_{i=1}^{n_k} B_{k,i}^{1/p'} \sigma_{k,i} \leq \eta$ and set $K = \bigcup_{k \leq l} Z_k$. Then, in the notation of the proof of Theorem 5.3, we have by Corollary 3.3, that

$$\begin{aligned} \varepsilon \# \mathcal{L}_2 &\leq \sum_{i,j} A(\Gamma_i^{*(j)}) \leq \sum_{i,j} \|T_{\Gamma_i^*(j)}\| \\ &\leq c \sum_{k \geq l} \sum_{i=1}^{n_k} B_{k,i}^{1/p'} \sigma_{k,i} \end{aligned}$$

for some positive constant c, by Lemma 6.3 with $q = 1$. The proofs of the first two identities then follow that of Theorem 5.3. Theorem 3.18 and Lemma 3.16 complete the proof.

Note that the convergence of $\sum_{k \in \mathbf{Z}} \sum_{i=1}^{n_k} B_{k,i}^{p'} \sigma_{k,i} < \infty$ implies that T is compact. To see this, let $K_j = \cup_{k \leq j} Z_k$, and so

$$\begin{aligned} (T_{K_j} f)(x) &= v(x) \chi_{K_j}(x) \int_a^x f(t) u(t) \chi_{K_j}(t) dt \\ &= v(x) \chi_{K_j}(x) \int_a^x f(t) u(t) dt. \end{aligned}$$

Then

$$(T - T_{K_j}) f(x) = v(x) \chi_{\Gamma \setminus K_j}(x) \int_a^x f(t) u(t) dt$$

and, by Lemma 6.2, for some $k' > j$

$$\begin{aligned}\|T - T_{K_j}\| &\leq 2^{2/p} c \left\{ \sum_{i \in S} 2^{k'} \mu(Z_{k',i}) \right\}^{1/p} \\ &\leq 2^{2/p} c \sum_{i \in S} \sigma_{k',i} \leq 2^{2/p} c \sum_{i \in S} B_{k',i}^{1/p'} \sigma_{k',i}.\end{aligned}$$

Thus $\|T - T_{K_j}\| \to 0$ as $j \to \infty$, and T is compact since the T_{K_j} are compact.□

Theorem 6.5 *Let $1 < q \leq p$. Then, for some positive constant c,*

$$\|\{a_n(T)\}\|_{l^q(\mathbf{N})} \leq c \|B_{k,i}^{1/p'} \sigma_{k,i}\|_{l^q(\mathbf{Z} \times \mathbf{N})} \text{ if } 1 < q \leq p \tag{6. 17}$$

Proof: Let $\Gamma_l, l = 1, 2, \ldots, M(\Gamma, \varepsilon)$ be a maximal set of subtrees of Γ from the definition of $M(\Gamma, \varepsilon)$, so that $A(\Gamma_l) > \varepsilon$. Then, from (6. 7) and Corollary 3.3, for $1 \leq q \leq p$,

$$\varepsilon^q M(\Gamma, \varepsilon) \leq \sum_l \|T_l\|^q \leq c \|B_{k,i}^{1/p'} \sigma_{k,i}\|^q_{l^q(\mathbf{Z} \times \mathbf{N})}.$$

Since $a_{3M(\Gamma,\varepsilon)+4}(T) \leq 2\varepsilon$ by Lemma 3.9 and Theorem 3.18, it follows that

$$\begin{aligned}\#\{m : a_m(T) > t\} &\leq 3M(\Gamma, t/2) + 4 \\ &\leq cM(\Gamma, t/2)\end{aligned}$$

for $c \geq 7$. Thus

$$\#\{m : a_m(T) > t\} \leq c t^{-q} \|B_{k,i}^{1/p'} \sigma_{k,i}\|^q_{l^q(\mathbf{Z} \times \mathbf{N})}. \tag{6. 18}$$

We now proceed as in the proof of the Marcinkiewicz Interpolation Theorem (see [11]); we give the proof for completeness.

Define

$$v_1 := \begin{cases} v & \text{on } Z_{k,i} \text{ if } B_{k,i}^{1/p'} \sigma_{k,i} \leq t/2, \\ 0 & \text{otherwise,} \end{cases}$$

and set $v_2 = v - v_1$. Denote $T, \sigma_{k,i}$, etc by $T(v), \sigma_{k,i}(v)$ to indicate the dependence on v, and set $T^j = T(v_j),\ j = 1, 2$. Then, by [3, Proposition II.2.2],

$$a_{2n-1}(T) \leq a_n(T_1) + a_n(T_2)$$

and so

$$\{n : a_{2n-1}(T) > t\} \subseteq \{n : a_n(T_1) > t/2\} \cup \{n : a_n(T_2) > t/2\}$$

and

$$\#\{n : a_{2n-1}(T) > t\} \le \#\{n : a_n(T_1) > t/2\} + \#\{n : a_n(T_2) > t/2\}. \quad (6.\ 19)$$

Set $S_{k,i} = B_{k,i}^{1/p'}\sigma_{k,i}$, and let $1 < q < q_1$. Then, on using (6. 18) and (6. 19),

$$\begin{aligned}
\|\{a_{2n-1}(T)\}\|_{l^q(\mathbf{N})} &= q\int_0^\infty t^{q-1}\#\{n : a_{2n-1}(T) > t\}dt \\
&\le cq\int_0^\infty t^{q-1}\left\{t^{-q_1}\sum_{S_{k,i}\le t/2} S_{k,i}^{q_1} + t^{-1}\sum_{S_{k,i}>t/2} S_{k,i}\right\}dt \\
&\le c\sum_{k,i} S_{k,i}^q,
\end{aligned}$$

whence (6. 17), since $a_n(T)$ decreases with n. □

Theorem 6.6 *Let $q \in (p,\infty)$. Then, for some positive constant c,*

$$\|\{a_n(T)\}\|_{l^q(\mathbf{N})} \le c\|\{\sigma_k\}\|_{l^q(\mathbf{Z})}. \quad (6.\ 20)$$

Proof. Let $\{\Gamma_l\}_1^{M(\Gamma,\varepsilon)}$ be as in the proof of Theorem 6.5 and define

$$F_j = \{\Gamma_l : k'(\Gamma_l) = k_j\}$$

in the notation of Lemma 6.2. Then, from the proof of Lemma 6.3, for $q \in (p,\infty)$,

$$\begin{aligned}
\varepsilon^p\#F_j &\le \sum_{\Gamma_l\in F_j}\|T_l\|^p \\
&\le c^p\sigma_{k_j}^p.
\end{aligned}$$

Thus, with $m_j = [c^p\sigma_{k_j}^p/\varepsilon^p]$,

$$M(\Gamma,\varepsilon) = \sum_{j=1}^\infty\sum_{m=1}^{m_j} 1 \le \sum_{m=1}^\infty \#\{j : c\sigma_{k_j} > m^{1/p}\varepsilon\}.$$

Hence

$$\begin{aligned}
\|\{a_n(T)\}\|_{l^q(\mathbf{N})} &= q\int_0^\infty t^{q-1}\#\{n : a_n(T) > t\}dt \\
&\le c\int_0^\infty t^{q-1}M(\Gamma,t/2)dt \\
&\le c\int_0^\infty \sum_{m=1}^\infty t^{q-1}\#\{j : \sigma_j > m^{1/p}t\}dt \\
&\le c\int_0^\infty \sum_{m=1}^\infty m^{-q/p}\#\{j : \sigma_j > t\}t^{q-1}dt \\
&\le c\|\{\sigma_k\}\|_{l^q(\mathbf{Z})}.
\end{aligned}$$

□

In the next theorem l^q_ω denotes weak-l^q, that is, the space of sequences $\{x_k\}$ such that

$$\|\{x_k\}\|_{l^q_\omega} := \sup_{t>0}\{t(\#\{k : |x_k| > t\})^{1/q}\} < \infty.$$

Theorem 6.7 *For some positive constant c,*

$$\|\{a_n(T)\}\|_{l^q_\omega(\mathbf{N})} \leq c\|\sum_{i=1}^{n_k} B_{k,i}^{1/p'}\sigma_{k,i}\|_{l^q_\omega(\mathbf{Z})} \text{ if } 1 < q \leq p, \tag{6. 21}$$

and

$$\|\{a_n(T)\}\|_{l^q_\omega(\mathbf{N})} \leq c\|\{\sigma_k\}\|_{l^q_\omega(\mathbf{Z})} \text{ if } p < q < \infty. \tag{6. 22}$$

Proof. Let $\{\Gamma_l\}_1^{M(\Gamma,\varepsilon)}, F_j$ be as in the proof of Theorem 6.6. Then, from the proof of Lemma 6.3,

$$\begin{aligned} \varepsilon\#F_j &\leq \sum_{\Gamma_l\in F_j}\|T_l\| \\ &\leq c\sum_{i=1}^{n_{k_j}} B_{k_j,i}^{1/p'}\sigma_{k_j,i} =: N_j \end{aligned}$$

say. Thus

$$\begin{aligned} \varepsilon^q\#\{n : a_n(T) > \varepsilon\} &\leq c\varepsilon^q M(\Gamma, \varepsilon/2) \\ &\leq c\varepsilon^q\sum_{j=1}^{\infty}\sum_{m=1}^{[N_j/\varepsilon]} 1 \\ &\leq c\varepsilon^q\sum_{m=1}^{\infty}\#\{j : N_j > m\varepsilon\} \\ &\leq c\sum_{m=1}^{\infty} m^{-q}t^q\#\{j : N_j > t\} \end{aligned}$$

and hence (6. 21). The proof of (6. 22) is similar, starting from

$$\varepsilon^p\#F_j \leq c\sigma_{k_j}^p.$$

□

Let us now suppose that the tree Γ satisfies the following condition:

$$B_{k,i} < B < \infty \quad \text{for each admissible } k \text{ and } i. \tag{6. 23}$$

Then with this condition we can get lower estimates in Theorems 6.5 and 6.6.

We need the following result which is similar to [2,Lemma 20].

Lemma 6.8 *Suppose that (6.10) is satisfied. Let $S(\varepsilon) := \{(k,i) : \sigma_{k,i} > \varepsilon\}$. If $M + 1/2 \leq \#S(\varepsilon)/4B$, then $a_M(T) > c\varepsilon$, where c is an absolute constant.*

Proof. It is sufficient to prove the result for $S(\varepsilon)$ finite, for this will imply the result when $\#S(\varepsilon) = \infty$. The elements of $S(\varepsilon)$ fall into two subsets according as k is odd or even. At least one of them, say $S_1(\varepsilon)$, has cardinal at least half that of $S(\varepsilon)$. Thus, we may suppose that $\#S_1(\varepsilon) > B$.

Denote by $\zeta_{k,j}$ the point of $Z_{k,j}$ nearest to a, and define $n(x) := \#\{(k,j) : \zeta_{k,j} \succ_a x, (k,j) \in S_1(\varepsilon)\}$. Let l be a path in Γ starting at a and consisting of edges (x,y) of Γ, (x,y) at each stage chosen so that $n(y)$ is as large as possible. Terminate the path at the point $x = \zeta_{r,s}$ at which $n(x) = 0$. Define $\xi \in l$ by

$$\xi := \inf\{x \in l : n(x) = n(\zeta_{r-1,j}), \zeta_{r-1,j} \in l\},$$

(the infimum, which is being taken with respect to the total ordering on l induced by $\preceq_a$, exists since $n(a) > B$ and $n(\zeta_{r-1,j}) \leq B$). There are two possibilities : (i) ξ may be a point $\zeta_{k,j}$, in which case define $\Gamma_1 := \{x : x \succ_a \xi\}$, or (ii) ξ may be a vertex of Γ joined by a path l_1 to a point $\zeta_{m,n} \succ_a \xi$, where $(m,n) \in S_1(\varepsilon)$. In the latter case we define $\Gamma_1 := \{x : x \succ_a y, y \in l \cup l_1\}$. Then, in both cases, the closure of Γ_1 is a subtree and so is its complement. Moreover, $A(\overline{\Gamma_1}) > c\varepsilon$, where c is an absolute constant. For, in case (i), if b is a point of l with $U(b) = 2^{(r-1/2)(p'/p)}$ and T^1 is the restriction of T to Γ_1, then, in the notation of the discussion preceding Lemma 3.5,

$$\|T^1_{b,1}\|, \|T^1_{b,2}\| \geq (2^{r(p'/p)} - 2^{(r-1/2)(p'/p)})^{1/p'}(2^{-r/p}\varepsilon);$$

this follows from [7, Proposition 5.1] where it is shown that

$$\|T_a\| \geq \sup_{x\in\Gamma} \|u\chi_(a,x)\|_{p'} \|v\chi_{(a,x)^c}\|_p.$$

In case (ii) a similar result holds if b is a point of $l \cup l_1$ with $U(b) = 2^{(t-1/2)(p'/p)}$ and t the greater of r, m. The lower bound for $A(\overline{\Gamma_1})$ is then a consequence of Lemma 3.5. Note also that Γ_1 contains at most $2B$ elements of $S_1(\varepsilon)$.

The result now follows by induction on $\#S(\varepsilon)$ and Lemma 3.17. □

Lemma 6.9 *Suppose that (6.10) is satisfied. Then, for all $t > 0$,*

$$\#\{(k,i) : \sigma_{k,i} > t\} \leq 4B\#\{k \in \mathbf{N} : a_k(T) > ct\} + 6B.$$

Proof. From Lemma 6.8,

$$\begin{aligned} \#\{k \in \mathbf{N} : a_k(T) > ct\} &\geq \left[\#S(t)/4B - 1/2\right] \\ &\geq \#S(t)/4B - 3/2, \end{aligned}$$

whence the result. □

Lemma 6.10 *Suppose that (6.10) is satisfied. Then, for all $q > 0$,*

$$\|\{\sigma_{k,i}\}\|^q_{l^q(\mathbf{Z}\times\mathbf{N})} \le c_1\|\{a_k(T)\}\|^q_{l^q(\mathbf{N})} + c_2\|\{\sigma_{k,i}\}\|^q_{l^\infty(\mathbf{Z}\times\mathbf{N})}$$

Proof. Let $\lambda = \|\{\sigma_{k,i}\}\|_{l^\infty(\mathbf{Z}\times\mathbf{N})}$. Then, by Lemma 6.9,

$$\begin{aligned}
\|\{\sigma_{k,i}\}\|^q_{l^q(\mathbf{Z}\times\mathbf{N})} &\le q\int_0^\lambda t^{q-1}\#\{(k,i)\in\mathbf{Z}\times\mathbf{N} : \sigma_{k,i} > t\}dt \\
&\le 4Bq\int_0^\lambda t^{q-1}\#\{k\in\mathbf{N}; a_k(T) > c\varepsilon\}dt + 6B\lambda^q \\
&\le c_1\|\{a_k(T)\}\|^q_{l^q(\mathbf{N})} + c_2\lambda^q.
\end{aligned}$$

□

Theorem 6.11 *Let $1 < p < \infty$ and suppose that (6.10) is satisfied. Then, for any $q > 0$, there exists a constant $c > 0$ such that*

$$\|\{\sigma_{k,i}\}\|_{l^q(\mathbf{Z}\times\mathbf{N})} \le c\|\{a_k(T)\}\|_{l^q(\mathbf{N})}.$$

Proof. By Lemma 6.1,

$$\|\{\sigma_{k,i}\}\|_{l^\infty(\mathbf{Z}\times\mathbf{N})} \le \|T\| = a_1(T) \le \|\{a_k(T)\}\|_{l^q(\mathbf{N})}.$$

The result then follows from Lemma 6.10. □

Remark 6.12 **(i)** *It follows from Theorem 6.5 and 6.11 that if (6. 23)is satisfied and $1 < q \le p$, then*

$$\|\{a_n(T)\}\|_{l^q(\mathbf{N})} \asymp \|\{\sigma_{k,i}\}\|_{l^q(\mathbf{Z}\times\mathbf{N})}.$$

For $q > p$, we have from Theorems 6.11 and 6.6

$$\begin{aligned}
c_1\|\sigma_{k_i}\|_{l^q(\mathbf{Z}\times\mathbf{N})} &\le \|\{a_n(T)\}\|_{l^q(\mathbf{N})} \le c_2\|\sigma_k\|_{l^q(\mathbf{Z})} \\
&\le \|\sigma_{k,i}\|_{l^p(\mathbf{Z}\times\mathbf{N})}. \qquad (6.\ 24)
\end{aligned}$$

(ii) *Naimark and Solomyak [8] take $u = 1$, and in [8,(4.8)] they make the assumption that, for every edge $\langle y, z\rangle \in E(\Gamma)$,*

$$\mu_1 \le |z|/|y| \le \mu_2, \qquad 1 < \mu_1 \le \mu_2, \qquad (6.\ 25)$$

where $|y|, |z|$ denote the lengths of the paths from the root of Γ to y, z respectively. Let $y_j \in V(t)$, $j = 0, 1, \dots,$ and suppose that $|y_0| \le 2^k$ and $|y_1| \ge 2^k$. Then (6. 25) implies that

$$|y_n| \ge \mu_1^{n-1}|y_1| \ge \mu_1^{n-1}2^k \ge 2^{k+1}$$

if $n \ge 1 + \log 2/\log\mu_1$. Hence, if each vertex has constant branching number b (ie. degree $b + 1$), then

$$B_{k,i} \le b^{[\log 2/log\mu_1+1]}$$

and hence (6. 23) is satisfied.

(iii) *Theorem 4.1 in [9] is valid under assumptions made on a sequence* $\{\eta_j\}$ *which is defined as follows : for any partition* Ξ *of* Γ *into a countable union of non-overlapping segments* $I_j = \langle y_j, z_j \rangle$,

$$\eta_j := |z_j| \int_{I_j} v^2 dt.$$

Note that in our notation, the case $p = 2, u = 1$ *is what is considered in [9]. It is proved in [9, Theorem 4.1 (i)] that (6.15) for* $p = 2$ *holds if, for some* Ξ, $\{\eta_j\} \in l_{1/2}$.

Choose $\Xi = \cup_{k \in \mathbf{N}} \Xi_k$, *where* Ξ_k *is a partition of* Z_k. *Then,*

$$\begin{aligned} \sigma_{k,j}^2 &= 2^k \sum_{I_s \subset Z_{k,j}} \int_{I_s} v^2 dt \leq \sum_{I_s \subset Z_{k,j}} \eta_s \\ &\leq \left(\sum_{I_s \subset Z_{k,j}} \eta_s^{1/2} \right)^2 \end{aligned}$$

and

$$\sum_{k,j} \sigma_{k,j} \leq \sum_s \eta_s^{1/2}.$$

Thus, if (6. 23) is satisfied,

$$\sum_{k,j} B_{k,j}^{1/2} \sigma_{k,j} \leq B^{1/2} \sum_s \eta_s^{1/2}. \tag{6. 26}$$

In the reverse directions we have

$$\sigma_{k,j}^2 \geq 1/2 \sum_{I_s \subset Z_{k,j}} \eta_s$$

and so

$$\sigma_{k,j} \geq 2^{-1/2} (\sum_{I_s \subset Z_{k,j}} \eta_s^{1/2}) (\sum_{I_s \subset Z_{k,j}} 1)^{-1/2}.$$

Therefore

$$B_{k,j}^{1/2} \sigma_{k,j} \geq 2^{-1/2} \left(\frac{B_{k,j}}{\sum_{I_s \subset Z_{k,j}} 1} \right)^{1/2} \sum_{I_s \subset Z_{k,j}} \eta_s^{1/2}.$$

If

$$\inf_{k,j} \left(\frac{B_{k,j}}{\sum_{I_s \subset Z_{k,j}} 1} \right) =: c > 0 \tag{6. 27}$$

then

$$\sum_{k,j} B_{k,j}^{1/2} \sigma_{k,j} \geq (c/2)^{1/2} \sum_s \eta_s^{1/2}. \tag{6. 28}$$

The condition (6. 27) is satisfied if the tree Γ *is, in the terminology of [9], b regular of type* $(b, 2)$ *and* Ξ *consists of edges of* Γ*. This means that every vertex of* Γ *has fixed branching number b, and any edge* $\langle y, z \rangle$ *of the k-th generation is such that* $|y| = 2^k, |z| = 2^{k+1}$*. Hence, in this case, (6. 27) is satisfied with* $c = 1$*.*

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